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Richardson Iterative Method for Solving Multi-Linear System with $\mathcal{M}$-Tensor

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Abstract

In this paper, Richardson iterative method is employed to solve $\mathcal{M}$-Equation. In order to guarantee the solution can be found, convergence theorems are established and confirmed numerically. The optimal α, which is a parameter of Richardson iterative method that can provide the best convergence rate, is also determined theoretically and numerically. Furthermore, a theorem establishing the range of initial vector for general splitting methods is extended from the range in past study. To further accelerate the convergence rate, Anderson accelerator and three preconditioners are incorporated into Richardson iterative method. Numerical results reveal that by including these accelerators, the convergence rates are enhanced. Finally, we show that Richardson iterative methods with optimal α perform better than the SOR type methods in past studies in terms of number of iterative steps and CPU time.

Keywords: Richardson iteration; $\mathcal{M}$-tensor; multi-linear system; Anderson acceleration; preconditioned technique.

1 Introduction

Multi-linear system, which is a generalization of linear system, has been gaining attentions due to its growing applications in various areas such as engineering [5], differential equations, data mining [10], data analysis [14] artificial intelligence [12], quantum mechanics [21], and economic [7]. Similar to a linear system, which can be expressed by matrix equation, a multi-linear system can be represented by tensor equation,

$$\mathcal{A}\mathbf{x}^{m-1} = \mathbf{b}, \quad \mathbf{x} \in \mathbf{V}^n, \tag{1}$$

where $\mathcal{A} = (a_{i_1 i_2 \dots i_m})$ is an m^{th} order and n dimensional tensor, $\mathbf{x} = (x_1, \dots, x_n)^T$ and $\mathbf{b} = (b_1, \dots, b_n)^T$ are n dimensional vectors, and $\mathbf{V}^n$ is an n dimensional real or complex vector space. The product of tensor and vector in Equation (1) is defined by

$$\mathbf{b}_i = \left(\mathcal{A}\mathbf{x}^{m-1}\right)_i = \sum_{i_2 \dots i_m = 1}^{n} a_{i i_2 \dots i_m} x_{i_2} \dots x_{i_m}, \quad i = 1, 2, \dots, n. \tag{2}$$

Equation (1) is called an $\mathcal{M}$-Equation if the coefficient tensor $\mathcal{A}$ is an $\mathcal{M}$-tensor, which is generalized from M matrix.

Ding and Wei [5] are the first to establish the existence of the solution for $\mathcal{M}$-Equation by proving that an $\mathcal{M}$-Equation has a unique positive solution if and only if $\mathcal{M}$-tensor is nonsingular and vector $\mathbf{b}$ is positive. Based on this finding, they developed the algorithms for solving $\mathcal{M}$-Equation using Jacobi type, Gauss-Seidel type, SOR type, and Newton iterative methods. Subsequently, Li et al. [8, 9] established the convergence theorems for the splitting iterative methods and their algorithms. Furthermore, they improved the convergence rate of the splitting methods by adding preconditioners.

However, the aforementioned studies did not consider finding the optimal parameter that can guarantee faster convergence rate. Recently, Pasini [16] obtained the optimal parameter of Richardson iterative method in solving linear system. More recently, Liang et al. [11] numerically solved third order $\mathcal{M}$-Equation using Richardson iterative method. Whether this method can be extended to improve the convergence rate in solving multi-linear systems remains an open question. Therefore, in this paper, we employ Richardson iterative method for solving $\mathcal{M}$-Equation.

The paper is organized as follows. Section 2 contains some preliminaries needed in this study. Then, a stationary Richardson iterative method for solving $\mathcal{M}$-Equation is proposed in Section 3. The convergence theorems and optimal parameter α of Richardson iterative method are established and determined theoretically. Furthermore, in this section, the range of initial vector for general splitting methods is extended from the range in past studies. In order to enhance the convergence rate, two strategies are employed; Anderson-type accelerator and preconditioner, as discussed in Sections 4 and 5, respectively. Numerical analysis comparisons between the above methods and SOR type methods in the previous studies in terms of convergence rate are illustrated in Section 6. Section 7 concludes this paper and provides possible future study.

2 Preliminaries

In this section, the definitions, theorems, and lemmas related to this study are presented.

Let $\mathcal{A} = (a_{i_1 i_2 \ldots i_m})$ be an mth order and n dimensional tensor. Then, $\mathcal{A}$ can be symmetrized on the last $(m-1)$ tubes by,

$$\mathcal{S} = (S_{i i_2 \ldots i_m}) = \left(\frac{1}{(m-1)!} \sum_{\tau \in \mathrm{T}}^{n} a_{i i_2 \ldots i_m} \right), \tag{3}$$

where T is the set of all the permutation of $(i_2, \ldots, i_m)$. Following Equation (2), the multi-linear system in Equation (1) can be rewritten as,

$$\mathcal{A}\mathbf{x}^{m-1} = \mathcal{S}\mathbf{x}^{m-1}, \qquad \mathbf{x} \in \mathbf{V}^n. \tag{4}$$

Therefore, without lost of generality, the coefficient tensor $\mathcal{A}$ is assumed to be symmetrized on the last $(m-1)$ tubes.

The following definition can be used to represent the product of matrix and tensor.

Definition 2.1. [9] *Let $\mathcal{A} \in \mathbb{C}^{[m,n]}$, matrix $B \in \mathbb{C}^{[2,n]}$. If $\mathcal{C} = B\mathcal{A}$, then,*

$$c_{i_1 i_2 \ldots i_m} = \sum_{j=1}^{n} b_{i_1 j} a_{j i_2 \ldots i_m}. \tag{5}$$

Note that the product of matrix and matrix is a special case of matrix-tensor product.

Below are some definitions related to $\mathcal{M}$-tensor.

Definition 2.2. [19] *Let $\mathcal{A} = (a_{i_1 i_2 \ldots i_m}) \in \mathbb{C}^{[m,n]}$. Its entries $a_{ii\ldots i}$, $1 \leqslant i \leqslant n$, are called diagonal entries and the rest are called off-diagonal entries. $\mathcal{A}$ is a diagonal tensor if and only if its off-diagonal entries are zero.*

Note that if diagonal tensor $\mathcal{A}$ is of second order, then it is reduced to a diagonal matrix.

Definition 2.3. [17] *Let $\mathcal{A} \in \mathbb{C}^{[m,n]}$. The majorization matrix $M(\mathcal{A})$ of $\mathcal{A}$ is a $n \times n$ matrix with $M(\mathcal{A})_{ij} = a_{ij\ldots j}$ for $i, j = 1, \ldots, n$.*

Definition 2.4. [13, 18] *Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$. A pair $(\lambda, \mathbf{x}) \in \mathbb{C} \times (\mathbb{C}^n \setminus 0)$ is an eigenpair (eigenvalue-eigenvector) of $\mathcal{A}$ if it satisfies,*

$$\mathcal{A}\mathbf{x}^{m-1} = \lambda \mathbf{x}^{[m-1]}, \tag{6}$$

where $\mathbf{x}^{[m-1]} = \left(x_1^{m-1}, \ldots, x_n^{m-1}\right)^T$. If $(\lambda, \mathbf{x}) \in \mathbb{R} \times (\mathbb{R}^n \setminus 0)$, then it is a H-eigenpair. The spectral radius of $\mathcal{A}$ is denoted by $\rho(\mathcal{A}) = \max\{|\lambda| \, | \lambda \in \sigma(\mathcal{A})\}$, where $\sigma(\mathcal{A})$ is the set of all eigenvalues of $\mathcal{A}$.

Definition 2.5. [4] *Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$. If the off-diagonal entries of $\mathcal{A}$ are nonpositive, then $\mathcal{A}$ is a $\mathcal{Z}$-tensor. $\mathcal{A}$ is called an $\mathcal{M}$-tensor if there exists a nonnegative tensor $\mathcal{B}$ and a positive real number $\eta \geqslant \rho(\mathcal{B})$ such that,*

$$\mathcal{A} = \eta \mathcal{I} - \mathcal{B}, \tag{7}$$

where $\mathcal{I}$ is the identity tensor. If $\eta > \rho(\mathcal{B})$, then $\mathcal{A}$ is called a strong $\mathcal{M}$-tensor.

The following are some properties of $\mathcal{M}$-tensor and spectral radius. Let $\mathbb{R}_+^{[m,n]}$ be the set of non-negative tensors with order m and dimension n.

Definition 2.6. [4] *$\mathcal{A}$ is a semi-positive tensor if and only if there exists $\mathbf{x} \geqslant 0$ such that $\mathcal{A}\mathbf{x}^{m-1} > 0$.*

Lemma 2.1. [4] *A $\mathcal{Z}$-tensor is a strong $\mathcal{M}$-tensor if and only if it is semi-positive.*

Lemma 2.2. [4] *A semi-positive $\mathcal{Z}$-tensor has all positive diagonal entries.*

Lemma 2.3. [1] *If $\mathcal{A} \in \mathbb{R}_{+}^{[m,n]}$, then there exist $\lambda \geqslant 0$ and a nonnegative vector $\mathbf{x} \neq 0$, such that,*

$$\mathcal{A}\mathbf{x}^{m-1} = \lambda \mathbf{x}^{[m-1]}. \tag{8}$$

Lemma 2.4. [9] *Let $\mathcal{A} \in \mathbb{R}_{+}^{[m,n]}$,*

i. If $\mu \mathbf{x}^{[m-1]} \leqslant (<) \mathcal{A}\mathbf{x}^{m-1}, \mathbf{x} \geqslant 0$ and $\mathbf{x} \neq 0$, then $\mu \leqslant (<) \rho(\mathcal{A})$.

ii. If $\mu \mathbf{x}^{[m-1]} \leqslant \mathcal{A}\mathbf{x}^{m-1}$ and $\mathbf{x} > 0$, then $\rho(\mathcal{A}) \leqslant \mu$.

For the existence and uniqueness of solution, Ding and Wei [5] obtained a sufficient condition for an $\mathcal{M}$-Equation to have a unique positive solution.

Lemma 2.5. [5] *If $\mathcal{A}$ is a nonsingular $\mathcal{M}$-tensor, then for every positive vector $\mathbf{b}$, the multi-linear system $\mathcal{A}\mathbf{x}^{m-1} = \mathbf{b}$ has a unique positive solution.*

The following theorem is necessary to discuss the convergence of iterative method.

Theorem 2.1. [20] *For operator $\phi(\mathbf{x}) : \mathbb{R}_n \to \mathbb{R}_n$, let $\phi(\mathbf{x}_*) = \mathbf{x}_*$ and $\nabla\phi : \mathbb{R}_n \to \mathbb{R}_{n\times n}$ be the Jacobian of ϕ. If $\sigma := \rho(\nabla\phi(\mathbf{x}_*)) < 1$, then $\mathbf{x}_*$ is an attracting fixed point of ϕ, which means there exist a real number $\delta > 0$ such that the sequence $\mathbf{x}_{k+1} = \phi(\mathbf{x}_k)$ is convergent when $\| \mathbf{x}_0 - \mathbf{x}_* \| \leqslant \delta$. Furthermore, if $\sigma > 0$, then the convergence is linear with rate σ.*

The properties of tensor splitting are necessary in order to prove a splitting iterative method is convergent.

Definition 2.7. [14] *Let $\mathcal{A}$, $\mathcal{E}$, and $\mathcal{F}$ be the same order and dimension tensors. $\mathcal{A} = \mathcal{E} - \mathcal{F}$ is*

i. a splitting of $\mathcal{A}$ if $\mathcal{E}$ is left-nonsingular,

ii. a regular splitting if $\mathcal{E}$ is left-nonsingular with $M(\mathcal{E})^{-1} \geqslant O$, and $\mathcal{F} \geqslant \mathcal{O}$,

iii. a weak regular splitting if $\mathcal{E}^{-1}$ is left-nonsingular with $M(\mathcal{E})^{-1} \geqslant O$, and $M(\mathcal{E})^{-1}\mathcal{F} \geqslant \mathcal{O}$,

iv. a convergent splitting if $\rho((\mathcal{E})^{-1}\mathcal{F}) < 1$.

Lemma 2.6. [14] *If $\mathcal{A}$ is a $\mathcal{Z}$-tensor, then the following statements are equivalent:*

i. $\mathcal{A}$ is a strong $\mathcal{M}$-tensor.

ii. $\mathcal{A}$ has a convergent (weak) regular splitting.

iii. All the (weak) regular splitting of $\mathcal{A}$ are convergent.

The comparison theorem for splitting iterative method is as follows;

Theorem 2.2. [9] *Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$ be an irreducible $\mathcal{Z}$-tensor and $\mathcal{A} = \mathcal{E}_1 - \mathcal{F}_1 = \mathcal{E}_2 - \mathcal{F}_2$ be a weak regular and regular splitting, respecitvely. If $\mathcal{F}_2 \leqslant \mathcal{F}_1$, $\mathcal{F}_2 \neq \mathcal{O}$, then,*

i. $\rho((\mathcal{E}_2)^{-1}\mathcal{F}_2) \leqslant \rho((\mathcal{E}_1)^{-1}\mathcal{F}_1) < 1$ *if and only if $\mathcal{A}$ is a strong $\mathcal{M}$-tensor.*

ii. $\rho((\mathcal{E}_2)^{-1}\mathcal{F}_2) = \rho((\mathcal{E}_1)^{-1}\mathcal{F}_1) = 1$ *if and only if $\mathcal{A}$ is a non-strong $\mathcal{M}$-tensor.*

iii. $\rho((\mathcal{E}_2)^{-1}\mathcal{F}_2) \geqslant \rho((\mathcal{E}_1)^{-1}\mathcal{F}_1) > 1$ *if and only if $\mathcal{A}$ is not an $\mathcal{M}$-tensor. In particular, if $\mathcal{F}_2 < \mathcal{F}_1$, $\mathcal{F}_2 \neq \mathcal{O}$, the first inequality is strict.*

In the next section, the stationary Richardson iterative method for linear system solution in [16] is extended to $\mathcal{M}$-Equation.

3 Stationary Richardson Iterative Method for $\mathcal{M}$-Equation Solution

The multi-linear system in (1) can be expressed as a fixed point problem

$$\mathcal{I}\mathbf{x}^{m-1} = (\mathcal{I} - \alpha\mathcal{A})\mathbf{x}^{m-1} + \alpha\mathbf{b}, \tag{9}$$

where $\alpha > 0$. The iterative scheme for (9) is

$$\mathcal{I}\mathbf{x}_{k+1}^{m-1} = \mathcal{I}\mathbf{x}_k^{m-1} + \alpha_k \left(\mathbf{b} - \mathcal{A}\mathbf{x}_k^{m-1}\right), \quad k = 0, 1, 2, \ldots, \tag{10}$$

also known as Richardson iterative method, where $\mathbf{x}_0$ is the initial guess and $\mathbf{b} - \mathcal{A}\mathbf{x}_k^{m-1}$ is the residual at the k-th iteration. When $\alpha_k = \alpha$ is fixed for every iteration, it is called stationary Richardson iteration.

Based on Lemma 2.5, an $\mathcal{M}$-Equation with strong $\mathcal{M}$-tensor and $\mathbf{b} > \mathbf{0}$ has unique positive solution. The Richardson iterative method for finding the unique positive solution of $\mathcal{M}$-Equation with strong $\mathcal{M}$-tensor and $\mathbf{b} > 0$, proposed by Liang el al. [11], is described in Algorithm 1.

Algorithm 1: Stationary Richardson Iterative Method

Input: Given a strong $\mathcal{M}$-tensor $\mathcal{A}$, a positive vector $\mathbf{b}$, an appropriate $\alpha > 0$, maximum iterative steps $k_{\max}$, tolerance number ε and a initial vector $\mathbf{x}_0 > \mathbf{0}$.

Output: $\mathbf{x}_k$

```
Set k = 1;
while k < k_max, do
    Set x_k = (I x_{k-1}^{m-1} + α (b − A x_{k-1}^{m-1}))^{[1/(m-1)]};
    if ||A x_k^{m-1} − b|| ⩽ ε then
        Output x_k;
        Stop;
    else
        k = k + 1;
    Output "The method fails since it exceeds k_max iterations."
```

In order to analyse the convergence of the stationary Richardson iterative method for solving $\mathcal{M}$-Equation as previously described, some properties of $\mathcal{M}$-tensor and M-matrix are provided.

Lemma 3.1. *Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$ be a strong $\mathcal{M}$-tensor. For any positive vector $\mathbf{x} \in \mathbb{R}^n$ with $\mathcal{A}\mathbf{x}^{m-1} > \mathbf{0}$, the matrix $\left(\operatorname{diag}(\mathbf{x}^{[m-2]})\right)^{-1} \mathcal{A}\mathbf{x}^{m-2}$ is a nonsingular* M-matrix.

Proof. Denote matrix $C = \left(\operatorname{diag}\left(\mathbf{x}^{[m-2]}\right)\right)^{-1} \mathcal{A}\mathbf{x}^{m-2}$. First we prove C is a Z-matrix.

By Lemma 2.1, since $\mathcal{A}$ is a strong $\mathcal{M}$-tensor, we have $\mathcal{A}$ is a semi-positive $\mathcal{Z}$-tensor. Then all the off-diagonal entries of $\mathcal{A}$ are nonpositive. Consider the off-diagonal entries of matrix C. The condition $\mathbf{x} > \mathbf{0}$ yields,

$$C_{ij} = \frac{1}{x_i^{m-2}} \sum_{i_3 \ldots i_m = 1}^{n} a_{iji_3 \ldots i_m} x_{i_3} \ldots x_{i_m} \leqslant 0,$$

where $i, j = 1, 2, \ldots, n$ and $i \neq j$. This shows that C is a Z-matrix.

Since $\mathcal{A}\mathbf{x}^{m-1} > \mathbf{0}$ and $\mathbf{x} > \mathbf{0}$, we have,

$$\begin{aligned} C\mathbf{x} &= \left[\left(\operatorname{diag}(\mathbf{x}^{[m-2]})\right)^{-1} \mathcal{A}\mathbf{x}^{m-2}\right] \mathbf{x} \\ &= \left(\operatorname{diag}(\mathbf{x}^{[m-2]})\right)^{-1} \left(\mathcal{A}\mathbf{x}^{m-2}\mathbf{x}\right) \\ &= \left(\operatorname{diag}(\mathbf{x}^{[m-2]})\right)^{-1} \left(\mathcal{A}\mathbf{x}^{m-1}\right) > \mathbf{0}. \end{aligned}$$

This means matrix C is semi-positive. Then C is a semi-positive Z-matrix, which is equivalent to C is a nonsingular M-matrix. □

Lemma 3.2. *Let $C \in \mathbb{R}^{n \times n}$ be a nonsingular* M-matrix, and $D = I - \alpha C$ where $\alpha \neq 0$ is a constant. $\sigma(C)$ and $\sigma(D)$ denote the set of all the eigenvalues of matrices C and D, respectively, while $\rho(D)$ is the spectral radius of D. If

$$\alpha \in \left(0, \min_{\lambda \in \sigma(C)} \frac{2 \operatorname{Re} \lambda}{|\lambda|^2}\right),$$

then $\rho(D) < 1$.

Proof. Since C is a nonsingular M-matrix, we have $\operatorname{Re} \lambda > 0$ for each $\lambda \in \sigma(C)$. Based on the properties of eigenvalue, $\lambda \in \sigma(C)$ if and only if $1 - \alpha\lambda \in \sigma(D)$. Then $\sigma(D) = \{1 - \alpha\lambda | \lambda \in \sigma(C)\}$ and $\rho(D) = \max_{\lambda \in \sigma(C)} \{|1 - \alpha\lambda|\}$. For $\lambda \in \sigma(C)$, to ensure $|1 - \alpha\lambda| < 1$, we square both sides of the inequality so that $|1 - \alpha\lambda|^2 < 1$. In fact,

$$\begin{aligned} |1 - \alpha\lambda|^2 &= (1 - \alpha \operatorname{Re} \lambda)^2 + (\alpha \operatorname{Im} \lambda)^2 \\ &= 1 - 2\alpha \operatorname{Re} \lambda + (\alpha \operatorname{Re} \lambda)^2 + (\alpha \operatorname{Im} \lambda)^2 \\ &= 1 - 2\alpha \operatorname{Re} \lambda + \alpha^2 |\lambda|^2 < 1, \end{aligned}$$

which can be rewritten as,

$$-2\alpha \operatorname{Re} \lambda + \alpha^2 |\lambda|^2 < 0. \tag{11}$$

Case 1: $\alpha > 0$,

If $\alpha > 0$, Equation (11) can be rewritten as $\alpha|\lambda|^2 < 2 \operatorname{Re} \lambda$. Thus, $0 < \alpha < \dfrac{2 \operatorname{Re} \lambda}{|\lambda|^2}$.

Case 2: $\alpha < 0$,

If $\alpha < 0$, Equation (11) can be rewritten as $\alpha|\lambda|^2 > 2\,\mathrm{Re}\,\lambda$. So $\alpha > \dfrac{2\,\mathrm{Re}\,\lambda}{|\lambda|^2} > 0$ since $\mathrm{Re}\,\lambda > 0$ for each $\lambda \in \sigma(C)$. This contradict with $\alpha < 0$. Thus, there is no α that meet the conditions in this case.

Therefore, if $\alpha \in \left(0, \min_{\lambda\in\sigma(C)} \dfrac{2\,\mathrm{Re}\,\lambda}{|\lambda|^2}\right)$, then $\rho(D) < 1$. □

Theorem 3.1 guarantees the stationary Richardson iterative for solving $\mathcal{M}$-Equation is convergent.

Theorem 3.1. *Let $\mathcal{A}\mathbf{x}^{m-1} = \mathbf{b}$ be a multi-linear system where $\mathcal{A} \in \mathbb{R}^{[m,n]}$ is a strong $\mathcal{M}$-tensor and $\mathbf{b} > \mathbf{0}$ with the exact solution $\mathbf{x}_* > \mathbf{0}$. The Richardson iteration,*

$$\mathcal{I}\mathbf{x}_{k+1}^{m-1} = \mathcal{I}\mathbf{x}_k^{m-1} + \alpha\left(\mathbf{b} - \mathcal{A}\mathbf{x}_k^{m-1}\right), \qquad k = 0, 1, 2, \ldots, \tag{12}$$

is convergent if

$$\alpha \in \left(0, \min_{\lambda\in\sigma(C)} \frac{2\,\mathrm{Re}\,\lambda}{|\lambda|^2}\right), \tag{13}$$

where $C = \left(\mathrm{diag}\left(\mathbf{x}_*^{[m-2]}\right)\right)^{-1} \mathcal{A}\mathbf{x}_*^{m-2}$ and $\sigma(C)$ is the set of all the eigenvalues of matrix C.

The iteration is locally linearly convergence with the rate,

$$\rho\left(\mathrm{diag}\left(\mathbf{x}_*^{[m-2]}\right)^{-1}\left(\mathcal{I} - \alpha\mathcal{A}\right)\mathbf{x}_*^{m-2}\right), \tag{14}$$

which is the spectral radius of matrix $\left(\mathrm{diag}\left(\mathbf{x}_*^{[m-2]}\right)\right)^{-1}\left(\mathcal{I} - \alpha\mathcal{A}\right)\mathbf{x}_*^{m-2}$.

Proof. Based on the Richardson iterative scheme, we define,

$$\phi(\mathbf{x})^{[m-1]} = \mathcal{I}\phi(\mathbf{x})^{m-1} = \left(\mathcal{I} - \alpha\mathcal{A}\right)\mathbf{x}^{m-1} + \alpha\mathbf{b}, \tag{15}$$

where $\phi(\mathbf{x}) : \mathbb{R}_n \to \mathbb{R}_n$, which leads to

$$\phi(\mathbf{x}) = \left(\left(\mathcal{I} - \alpha\mathcal{A}\right)\mathbf{x}^{m-1} + \alpha\mathbf{b}\right)^{\left[\frac{1}{m-1}\right]}. \tag{16}$$

Since $\mathcal{A}$ is symmetric on the last $(m-1)$ tubes, then

$$\nabla(\mathcal{A}\mathbf{x}^{m-1}) = (m-1)\mathcal{A}\mathbf{x}^{m-2},$$

where $\mathcal{A}\mathbf{x}^{m-2}$ is an $n \times n$ matrix with

$$\left(\mathcal{A}\mathbf{x}^{m-2}\right)_{ij} = \sum_{i_3\ldots i_m=1}^{n} a_{iji_3\ldots i_m} x_{i_3}\ldots x_{i_m}, \qquad i, j = 1, 2, \ldots, n.$$

Taking the gradients on both sides of (15) produces

$$\mathcal{I}\phi(\mathbf{x})^{m-2}\nabla(\phi(\mathbf{x})) = \left(\mathcal{I} - \alpha\mathcal{A}\right)\mathbf{x}^{m-2}.$$

Since $\phi(\mathbf{x}_*) = \mathbf{x}_*$, the Jacobian of $\phi(\mathbf{x})$ at $\mathbf{x}_*$ is

$$\begin{aligned}\nabla\left(\phi\left(\mathbf{x}_*\right)\right) &= \left(\mathcal{I}\phi\left(\mathbf{x}_*\right)^{m-2}\right)^{-1}\left(\mathcal{I}-\alpha\mathcal{A}\right)\mathbf{x}_*^{m-2}\\ &= \left(\mathcal{I}\mathbf{x}_*^{m-2}\right)^{-1}\left(\mathcal{I}-\alpha\mathcal{A}\right)\mathbf{x}_*^{m-2}\\ &= \left(\operatorname{diag}\left(\mathbf{x}_*^{[m-2]}\right)\right)^{-1}\left(\mathcal{I}-\alpha\mathcal{A}\right)\mathbf{x}_*^{m-2}\\ &= I-\alpha\left(\operatorname{diag}\left(\mathbf{x}_*^{[m-2]}\right)\right)^{-1}\mathcal{A}\mathbf{x}_*^{m-2}. \end{aligned} \tag{17}$$

Based on Theorem 2.1, if $\rho\left(\nabla\left(\phi\left(\mathbf{x}_*\right)\right)\right) < 1$, then the iteration (12) is convergent.

Let $C = \left(\operatorname{diag}\left(\mathbf{x}_*^{[m-2]}\right)\right)^{-1}\mathcal{A}\mathbf{x}_*^{m-2}$. Then Equation (17) can be simplified to $\nabla\left(\phi\left(\mathbf{x}_*\right)\right) = I-\alpha C$. Let $\sigma(\nabla\left(\phi\left(\mathbf{x}_*\right)\right)) = \left\{1-\alpha\lambda|\lambda\in\sigma(C)\right\}$ represents the set of eigenvalues. Then, $\rho(\nabla\left(\phi\left(\mathbf{x}_*\right)\right)) = \max\limits_{\lambda\in\sigma(C)}\left\{|1-\alpha\lambda|\right\}$.

Based on Lemma 3.1, it can be concluded that C is a nonsingular M-matrix. According to Lemma 3.2, if

$$\alpha\in\left(0,\min_{\lambda\in\sigma(C)}\frac{2\operatorname{Re}\lambda}{|\lambda|^2}\right),$$

we have $\rho\left(\nabla\left(\phi\left(\mathbf{x}_*\right)\right)\right) < 1$. Therefore, by Theorem 2.1, the Richardson iteration is locally linearly convergent with the rate,

$$\rho\left(\operatorname{diag}\left(\mathbf{x}_*^{[m-2]}\right)^{-1}\left(\mathcal{I}-\alpha\mathcal{A}\right)\mathbf{x}_*^{m-2}\right),$$

which is the spectral radius of matrix $\left(\operatorname{diag}\left(\mathbf{x}_*^{[m-2]}\right)\right)^{-1}\left(\mathcal{I}-\alpha\mathcal{A}\right)\mathbf{x}_*^{m-2}$. □

If the matrix $\left(\operatorname{diag}\left(\mathbf{x}_*^{[m-2]}\right)\right)^{-1}\mathcal{A}\mathbf{x}_*^{m-2}$ is not only an nonsingular M-matrix, but also strictly diagonally dominant, then we can have the following corollary.

Corollary 3.1. *Let $\mathcal{A}\mathbf{x}^{m-1} = \mathbf{b}$ be a multi-linear system where $\mathcal{A}\in\mathbb{R}^{[m,n]}$ is a strong $\mathcal{M}$-tensor and $\mathbf{b} > \mathbf{0}$ with the exact solution $\mathbf{x}_* > \mathbf{0}$. Assume $C = \left(\operatorname{diag}\left(\mathbf{x}_*^{[m-2]}\right)\right)^{-1}\mathcal{A}\mathbf{x}_*^{m-2}$ is strictly diagonally dominant. Then the Richardson iteration in Equation (12) is convergent if*

$$\alpha\in\left(0,\frac{2}{\rho(C)}\right),$$

where $\rho(C)$ is the spectral radius of matrix C.

Proof. By the assumption $C = \left(\operatorname{diag}\left(\mathbf{x}_*^{[m-2]}\right)\right)^{-1}\mathcal{A}\mathbf{x}_*^{m-2}$ is strictly diagonally dominant and Lemma 3.1, C is a nonsingular strictly diagonally dominant M-matrix. Then C is positive definite, which indicates that all the eigenvalues of C are positive, and $\rho(C) = \max\limits_{\lambda\in\sigma(C)}\{\lambda\}$. Therefore, $\min\limits_{\lambda\in\sigma(C)}\frac{2\operatorname{Re}\lambda}{|\lambda|^2} = \frac{2}{\rho(C)}$. Thus, by Theorem 3.1, the iteration (12) is convergent if,

$$\alpha\in\left(0,\min_{\lambda\in\sigma(C)}\frac{2\operatorname{Re}\lambda}{|\lambda|^2}\right) = \left(0,\frac{2}{\rho(C)}\right).$$

□

According to Theorem 3.1, the Richardson iteration is convergent with the rate $\rho\left(\operatorname{diag}\left(\mathbf{x}_*^{[m-2]}\right)^{-1}(\mathcal{I}-\alpha\mathcal{A})\,\mathbf{x}_*^{m-2}\right)$. Since smaller iteration convergence rate implies faster convergent, this motivates us to determine the optimal α that can guarantee the best convergence rate.

Theorem 3.2. *Let $\mathcal{A}\mathbf{x}^{m-1}=\mathbf{b}$ be a multi-linear system where $\mathcal{A}\in\mathbb{R}^{[m,n]}$ is a strong $\mathcal{M}$-tensor and $\mathbf{b}>\mathbf{0}$ with the exact solution $\mathbf{x}_*>\mathbf{0}$. Let $C=\left(\operatorname{diag}\left(\mathbf{x}_*^{[m-2]}\right)\right)^{-1}\mathcal{A}\mathbf{x}_*^{m-2}$ and $\sigma(C)=\{\lambda|\lambda \text{ is eigenvalue of } C\}$. The Richardson iteration in Equation (12) for solving $\mathcal{A}\mathbf{x}^{m-1}=\mathbf{b}$ achieves the minimal convergence rate when,*

$$\alpha=\alpha_{opt}=\begin{cases}\dfrac{2(\operatorname{Re}\lambda_1-\operatorname{Re}\lambda_2)}{|\lambda_1|^2-|\lambda_2|^2}, & \lambda_1\neq\lambda_2,\\[2ex] \dfrac{1}{\lambda_1}, & \lambda_1=\lambda_2,\end{cases}\tag{18}$$

where,

$$\lambda_1=\arg\max_{\lambda\in\sigma(C)}\frac{\operatorname{Re}\lambda}{|\lambda|^2},\quad \textit{and}\quad \lambda_2=\arg\min_{\lambda\in\sigma(C)}\frac{\operatorname{Re}\lambda}{|\lambda|^2}.\tag{19}$$

Proof. According to Theorem 3.1, if $\alpha\in\left(0,\min\limits_{\lambda\in\sigma(C)}\dfrac{2\operatorname{Re}\lambda}{|\lambda|^2}\right)$, the convergence rate of Richardson iteration in Equation (12) is

$$\rho\left(\operatorname{diag}\left(\mathbf{x}_*^{[m-2]}\right)^{-1}(\mathcal{I}-\alpha\mathcal{A})\,\mathbf{x}_*^{m-2}\right)=\rho(I-\alpha C)<1.\tag{20}$$

Since α is undetermined, the convergence rate (20) can be regarded as a function of α, i.e., $\rho(\alpha)=\max\limits_{\lambda\in\sigma(C)}\{|1-\alpha\lambda|\}$. So,

$$\begin{aligned}\alpha_{opt}&=\arg\min_{\alpha}\rho(\alpha)\\&=\arg\min_{\alpha}\max_{\lambda\in\sigma(C)}\{|1-\alpha\lambda|\}\\&=\arg\min_{\alpha}\max_{\lambda\in\sigma(C)}\{|1-\alpha\lambda|^2\}.\end{aligned}$$

In order to obtain α_{opt}, we denote $f_\lambda(\alpha)=|1-\alpha\lambda|^2$ for any $\lambda\in\sigma(C)$. Then,

$$\alpha_{opt}=\arg\min_{\alpha}\max_{\lambda\in\sigma(C)}\{|1-\alpha\lambda|^2\}=\arg\min_{\alpha}\max_{\lambda\in\sigma(C)}f_\lambda(\alpha)=\arg\min_{\alpha}\rho^2(\alpha),\tag{21}$$

and

$$\begin{aligned}f_\lambda(\alpha)=|1-\alpha\lambda|^2&=(1-\alpha\operatorname{Re}\lambda)^2+(\alpha\operatorname{Im}\lambda)^2\\&=1-2\alpha\operatorname{Re}\lambda+(\alpha\operatorname{Re}\lambda)^2+(\alpha\operatorname{Im}\lambda)^2\\&=1-2\alpha\operatorname{Re}\lambda+\alpha^2|\lambda|^2.\end{aligned}\tag{22}$$

We can see that $f_\lambda(\alpha)$ is a quadratic function with independent variable α and quadratic coefficient $|\lambda|^2>0$. According to Lemma 3.1, matrix C is a nonsingular M-matrix, which indicates $\operatorname{Re}\lambda>0$.

Then the symmetry axis of quadratic function $f_\lambda(\alpha)$ is $L = \dfrac{\operatorname{Re}\lambda}{|\lambda|^2} > 0$. Also, $f_\lambda(\alpha)$ has the minimal value $1 - \left(\dfrac{\operatorname{Re}\lambda}{|\lambda|}\right)^2 = \left(\dfrac{\operatorname{Im}\lambda}{|\lambda|}\right)^2 > 0$.

Figure 1 shows $f_\lambda(\alpha)$ for three different eigenvalues; the purple parabola represents $f_{\lambda_1}(\alpha)$ where the symmetry axis is farthest from y-axis with $\lambda_1 = \arg\max\limits_{\lambda\in\sigma(C)} \dfrac{\operatorname{Re}\lambda}{|\lambda|^2}$, the blue parabola represents $f_{\lambda_2}(\alpha)$ where the symmetry axis is nearest to y-axis with $\lambda_2 = \arg\min\limits_{\lambda\in\sigma(C)} \dfrac{\operatorname{Re}\lambda}{|\lambda|^2}$, and the green parabola represents $f_\lambda(\alpha)$ where the symmetry axis lies between the one of $f_{\lambda_1}(\alpha)$ and $f_{\lambda_2}(\alpha)$. Since $\rho(\alpha) = \max\limits_{\lambda\in\sigma(C)}\{|1-\alpha\lambda|\} < 1$, we only need to consider the graph when $f_\lambda(\alpha) < 1$ for every $\lambda \in \sigma(C)$. Thus, the graph of function $\rho^2(\alpha) = \max\limits_{\lambda\in\sigma(C)} f_\lambda(\alpha)$ is shown by a bold line in Figure 1.

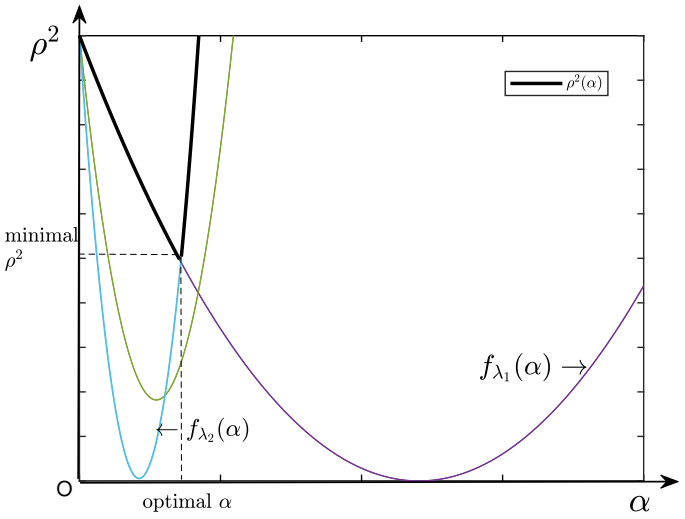


Figure 1: Graph of $\rho^2(\alpha) = \max\limits_{\lambda\in\sigma(C)} f_\lambda(\alpha)$.

From Figure 1, we can see that the bold line coincides with $f_{\lambda_1}(\alpha)$ when $\alpha \in (0, \alpha_{opt})$, and coincides with $f_{\lambda_2}(\alpha)$ when $\alpha \in \left(\alpha_{opt}, \min\limits_{\lambda\in\sigma(C)} \dfrac{2\operatorname{Re}\lambda}{|\lambda|^2}\right)$. Consequently, the optimal α is the coordinate of the first instance of intersection between $f_{\lambda_1}(\alpha)$ and $f_{\lambda_2}(\alpha)$ occurs. Therefore, by letting $f_{\lambda_1}(\alpha) = f_{\lambda_2}(\alpha)$,

$$\begin{aligned} 1 - 2\alpha\operatorname{Re}\lambda_1 + \alpha^2|\lambda_1|^2 &= 1 - 2\alpha\operatorname{Re}\lambda_2 + \alpha^2|\lambda_2|^2 \\ \alpha\Big(|\lambda_1|^2 - |\lambda_2|^2\Big) &= 2\Big(\operatorname{Re}\lambda_1 - \operatorname{Re}\lambda_2\Big). \end{aligned} \tag{23}$$

If $\lambda_1 = \lambda_2$, then all the eigenvalues of C are the same real number. So $\alpha_{opt} = \dfrac{\operatorname{Re}\lambda_1}{|\lambda|_1^2} = \dfrac{1}{\lambda_1}$.

If $\lambda_1 \neq \lambda_2$, then Equation (23) can be rewritten as $\alpha = \alpha_{opt} = \dfrac{2(\operatorname{Re}\lambda_1 - \operatorname{Re}\lambda_2)}{|\lambda_1|^2 - |\lambda_2|^2}$. □

As previously mentioned before Corollary 3.1, all the eigenvalues of C are positive. Consequently, by Theorem 3.2, we have the following corollary.

Corollary 3.2. *Let $\mathcal{A}\mathbf{x}^{m-1} = \mathbf{b}$ be a multi-linear system where $\mathcal{A} \in \mathbb{R}^{[m,n]}$ is a strong $\mathcal{M}$-tensor and $\mathbf{b} > \mathbf{0}$ with the exact solution $\mathbf{x}_* > \mathbf{0}$, $C = \left(\text{diag}\left(\mathbf{x}_*^{[m-2]}\right)\right)^{-1} \mathcal{A}\mathbf{x}_*^{m-2}$, and $\sigma(C) = \{\lambda | \lambda \text{ is eigenvalue of } C\}$. Assume $C = \left(\text{diag}\left(\mathbf{x}_*^{[m-2]}\right)\right)^{-1} \mathcal{A}\mathbf{x}_*^{m-2}$ is strictly diagonally dominant. The Richardson iteration in Equation (12) for solving $\mathcal{A}\mathbf{x}^{m-1} = \mathbf{b}$ achieves the minimal convergence rate when*

$$\alpha = \alpha_{opt} = \frac{2}{\lambda_1 + \lambda_2}, \tag{24}$$

where,

$$\lambda_1 = \min_{\lambda \in \sigma(C)} \{\lambda\}, \qquad \text{and} \qquad \lambda_2 = \max_{\lambda \in \sigma(C)} \{\lambda\}.$$

The minimal convergence rate is

$$\rho(\alpha_{opt}) = \frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2}. \tag{25}$$

Proof. Since all the eigenvalues of C are positive, we can obtain α_{opt} in (24) from Equation (18), and calculate $\rho(\alpha_{opt})$ in (25) by evaluating $\rho(\alpha) = \max_{\lambda \in \sigma(C)} \{|1 - \alpha\lambda|\}$ at α_{opt} in (24). □

Remark 3.1. *Theorem 3.1 and Corollary 3.1 provided the area of α for convergent Richardson iteration in solving $\mathcal{M}$-Equation. Theorem 3.2 and Corollary 3.2 give the optimal α to achieve minimal convergence rate. Note that these results are also valid when applied to Richardson iteration for solving linear systems. The proofs are similar to the proof in the above theorems and corollaries.*

The convergence area and optimal value discussed in above theorems and corollaries involved parameter α that depends on the exact solution $\mathbf{x}_*$. In the case when $\mathbf{x}_*$ is unknown, the following theorem is necessary.

Theorem 3.3. *Let $\mathcal{A}\mathbf{x}^{m-1} = \mathbf{b}$ be a multi-linear system where $\mathcal{A} \in \mathbb{R}^{[m,n]}$ is an $\mathcal{M}$-tensor and $\mathbf{b} > \mathbf{0}$. The stationary Richardson iteration is locally convergent if*

$$\alpha \in \left(0, \frac{1}{\max\limits_{i \in <n>} a_{ii\cdots i}}\right]. \tag{26}$$

Proof. Let $\mathbf{x}_*$ be the exact positive solution of $\mathcal{M}$-Equation $\mathcal{A}\mathbf{x}^{m-1} = \mathbf{b}$. According to Theorem 3.1, the Richardson iteration is convergent if

$$\rho\left(\nabla\left(\phi\left(\mathbf{x}_*\right)\right)\right) = \rho\left(\text{diag}\left(\mathbf{x}_*^{[m-2]}\right)^{-1}\left(\mathcal{I} - \alpha\mathcal{A}\right)\mathbf{x}_*^{m-2}\right) < 1.$$

Let $\alpha \in \left(0, \frac{1}{\max\limits_{i \in <n>} a_{ii\cdots i}}\right]$. Since $\mathcal{A}$ is an $\mathcal{M}$-tensor, $\mathcal{I} - \alpha\mathcal{A}$ is a nonnegative tensor. Then matrix

$\left(\text{diag}\left(\mathbf{x}_*^{[m-2]}\right)\right)^{-1}(\mathcal{I}-\alpha\mathcal{A})\,\mathbf{x}_*^{m-2}$ is also nonnegative. Multiply it with $\mathbf{x}_*$, we have

$$\begin{aligned}
&\left(\text{diag}\left(\mathbf{x}_*^{[m-2]}\right)\right)^{-1}(\mathcal{I}-\alpha\mathcal{A})\,\mathbf{x}_*^{m-2}\mathbf{x}_* \\
&= \left(\text{diag}\left(\mathbf{x}_*^{[m-2]}\right)\right)^{-1}\mathbf{x}_*^{[m-1]} - \alpha\left(\text{diag}\left(\mathbf{x}_*^{[m-2]}\right)\right)^{-1}\mathcal{A}\mathbf{x}_*^{m-1} \\
&= \mathbf{x}_* - \alpha\left(\text{diag}\left(\mathbf{x}_*^{[m-2]}\right)\right)^{-1}\mathbf{b} < \mathbf{x}_*. \qquad (27)
\end{aligned}$$

Applying the minimax theorem of nonnegative matrices from [6], we get

$$\begin{aligned}
&\rho\left(\left(\text{diag}\left(\mathbf{x}_*^{[m-2]}\right)\right)^{-1}(\mathcal{I}-\alpha\mathcal{A})\,\mathbf{x}_*^{m-2}\right) \\
&= \min_{\mathbf{x}\geqslant 0, x\neq 0}\max_{x_i>0}\frac{\left(\left(\text{diag}\left(\mathbf{x}_*^{[m-2]}\right)\right)^{-1}(\mathcal{I}-\alpha\mathcal{A})\,\mathbf{x}_*^{m-2}\mathbf{x}\right)_i}{x_i} \\
&\leqslant \max_{\mathbf{x}_{*,i}>0}\frac{\left(\left(\text{diag}\left(\mathbf{x}_*^{[m-2]}\right)\right)^{-1}(\mathcal{I}-\alpha\mathcal{A})\,\mathbf{x}_*^{m-2}\mathbf{x}_*\right)_i}{x_{*,i}} \\
&= \max_{\mathbf{x}_{*,i}>0}\frac{\left(\mathbf{x}_* - \alpha\left(\text{diag}\left(\mathbf{x}_*^{[m-2]}\right)\right)^{-1}\mathbf{b}\right)_i}{x_{*,i}} < 1.
\end{aligned}$$

This means the Richardson iteration is convergent. □

Theorem 3.3 shows the condition for choosing α only depends on the element of $\mathcal{A}$ that can guarantees the convergence of the Richardson iteration. Since the Richardson iteration is locally convergent, the initial vector $\mathbf{x}_0$ must be determined. First, we provide a general theorem for all splitting methods, before giving out the corollary for Richardson iterative method.

Theorem 3.4. *Let $\mathcal{A}\mathbf{x}^{m-1} = \mathbf{b}$ be a multi-linear system where $\mathcal{A} \in \mathbb{R}^{[m,n]}$ is a strong $\mathcal{M}$-tensor and $\mathbf{b} > \mathbf{0}$. If $\mathcal{A} = \mathcal{E} - \mathcal{F}$ is a weak regular splitting of $\mathcal{A}$, then the iteration,*

$$\mathbf{x}_k = \left(M(\mathcal{E})^{-1}\mathcal{F}\mathbf{x}_{k-1}^{m-1} + M(\mathcal{E})^{-1}\mathbf{b}\right)^{\left[\frac{1}{m-1}\right]},\ k = 1, 2, \ldots \qquad (28)$$

is locally convergent for any positive initial vector $\mathbf{x}_0$ that satisfies $\mathcal{A}\mathbf{x}_0^{m-1} > \mathbf{0}$.

Proof. Based on Lemma 2.1, there exists $\mathbf{x}_0 > \mathbf{0}$ such that $\mathcal{A}\mathbf{x}_0^{m-1} > \mathbf{0}$ since $\mathcal{A}$ is a strong $\mathcal{M}$-tensor. Assume that $\mathcal{A}\mathbf{x}_0^{m-1} \neq \mathbf{b}$, otherwise $\mathbf{x}_0$ is the solution.

Case 1: $\mathcal{A}\mathbf{x}_0^{m-1} > \mathbf{b}$

First, we prove the sequence $\{\mathbf{x}_k\}$ generated by Equation (28) is a decreasing sequence in this case. For $k = 1$, Equation (28) becomes

$$\begin{aligned}
\mathbf{x}_1^{[m-1]} &= M(\mathcal{E})^{-1}\mathcal{F}\mathbf{x}_0^{m-1} + M(\mathcal{E})^{-1}\mathbf{b} \\
&= M(\mathcal{E})^{-1}(\mathcal{E}-\mathcal{A})\mathbf{x}_0^{m-1} + M(\mathcal{E})^{-1}\mathbf{b} \\
&= \left(\mathcal{I} - M(\mathcal{E})^{-1}\mathcal{A}\right)\mathbf{x}_0^{m-1} + M(\mathcal{E})^{-1}\mathbf{b} \qquad (29) \\
&= \mathbf{x}_0^{[m-1]} + M(\mathcal{E})^{-1}\left(\mathbf{b} - \mathcal{A}\mathbf{x}_0^{m-1}\right). \qquad (30)
\end{aligned}$$

Since $\mathcal{A} = \mathcal{E} - \mathcal{F}$ is a weak regular splitting of $\mathcal{A}$, it has $M(\mathcal{E})^{-1} \geqslant O$ and $M(\mathcal{E})^{-1}\mathcal{F} \geqslant \mathcal{O}$. Given that $\mathcal{A}\mathbf{x}_0^{m-1} > \mathbf{b}$, we have $M(\mathcal{E})^{-1}\left(\mathbf{b} - \mathcal{A}\mathbf{x}_0^{m-1}\right) < \mathbf{0}$. So from Equation (30), $\mathbf{x}_0 > \mathbf{x}_1$.

Suppose that $\mathbf{x}_k < \mathbf{x}_{k-1}$. From Equation (29),

$$\begin{aligned} \mathcal{I} - M(\mathcal{E})^{-1}\mathcal{A} &= \mathcal{I} - M(\mathcal{E})^{-1}(\mathcal{E} - \mathcal{F}) \\ &= \mathcal{I} - M(\mathcal{E})^{-1}\mathcal{E} + M(\mathcal{E})^{-1}\mathcal{F}. \end{aligned} \tag{31}$$

According to the definition of majorization matrix of tensor in Definition 2.3, we have $\mathcal{I} - M(\mathcal{E})^{-1}\mathcal{E} \geqslant \mathcal{O}$. Because $M(\mathcal{E})^{-1}\mathcal{F} \geqslant \mathcal{O}$, $\mathcal{I} - M(\mathcal{E})^{-1}\mathcal{A}$ is a positive tensor. Thus,

$$(\mathcal{I} - M(\mathcal{E})^{-1}\mathcal{A})\mathbf{x}_k^{m-1} < (\mathcal{I} - M(\mathcal{E})^{-1}\mathcal{A})\mathbf{x}_{k-1}^{m-1},$$

and similarly, we have

$$\begin{aligned} \mathbf{x}_{k+1}^{[m-1]} &= (\mathcal{I} - M(\mathcal{E})^{-1}\mathcal{A})\mathbf{x}_k^{m-1} + M(\mathcal{E})^{-1}\mathbf{b} \\ &< (\mathcal{I} - M(\mathcal{E})^{-1})\mathbf{x}_{k-1}^{m-1} + M(\mathcal{E})^{-1}\mathbf{b} = \mathbf{x}_k^{[m-1]}. \end{aligned} \tag{32}$$

This means the sequence $\{\mathbf{x}_k\}$ is decreasing.

Secondly, we show $\{\mathbf{x}_k\}$ has positive lower bound. From Equation (32),

$$\begin{aligned} \mathbf{x}_k^{[m-1]} &= (\mathcal{I} - M(\mathcal{E})^{-1}\mathcal{A})\mathbf{x}_{k-1}^{m-1} + M(\mathcal{E})^{-1}\mathbf{b} \\ &> M(\mathcal{E})^{-1}\mathbf{b} > \mathbf{0}, \text{ for } k = 1, 2, \ldots. \end{aligned} \tag{33}$$

Then $M(\mathcal{E})^{-1}\mathbf{b}$ is a positive lower bound for the sequence $\{\mathbf{x}_k\}$. Therefore, the iteration of Equation (28) is convergent.

Case 2: If **Case 1** is not satisfied, set

$$\tau > \max_{i \in <n>} \left(\frac{\mathbf{b}_i}{\left(\mathcal{A}\mathbf{x}_0^{m-1}\right)_i} \right)^{\frac{1}{m-1}}, \text{ and } \mathbf{y}_0 = \tau \mathbf{x}_0.$$

Then, $\mathcal{A}\mathbf{y}_0^{m-1} = \tau^{m-1}\mathcal{A}\mathbf{x}_0^{m-1} > \mathbf{b}$, which is actually **Case 1**.

□

Remark 3.2. *$\mathcal{A}\mathbf{x}_0^{m-1} > \mathbf{0}$ is only a sufficient but not necessary condition for convergence. In [14], they prove that similar result is valid for $\boldsymbol{0} < \mathcal{A}\mathbf{x}_0^{m-1} < \boldsymbol{b}$ only, while in this paper, we extend to $\mathcal{A}\mathbf{x}_0^{m-1} > \boldsymbol{0}$ in Theorem 3.4.*

A corollary for Richardson iteration to Theorem 3.4 is as follows;

Corollary 3.3. *Let $\mathcal{A}\mathbf{x}^{m-1} = \mathbf{b}$ be a multi-linear system where $\mathcal{A} \in \mathbb{R}^{[m,n]}$ is a strong $\mathcal{M}$-tensor and $\mathbf{b} > \mathbf{0}$. The Richardson iteration with parameter $\alpha \in \left(0, \frac{1}{\max\limits_{i \in <n>} a_{ii\cdots i}}\right]$ is locally convergent for any positive initial vector $\mathbf{x}_0$ that satisfies $\mathcal{A}\mathbf{x}_0^{m-1} > \mathbf{0}$.*

Proof. Since $\mathcal{I} - \alpha\mathcal{A}$ is nonnegative with $\alpha \in \left(0, \frac{1}{\max\limits_{i} a_{ii\cdots i}}\right]$, and $\mathcal{I}^{-1} = \mathcal{I} > \mathcal{O}$, we have $\alpha\mathcal{A} = \mathcal{I} - (\mathcal{I} - \alpha\mathcal{A})$ is a weak regular splitting for $\alpha\mathcal{A}$. Based on Theorem 3.4, the Richardson iteration with parameter $\alpha \in \left(0, \frac{1}{\max\limits_{i \in <n>} a_{ii\cdots i}}\right]$ is locally convergent for any positive initial vector $\mathbf{x}_0$ that satisfies $\mathcal{A}\mathbf{x}_0^{m-1} > \mathbf{0}$. □

4 Anderson Acceleration of Richardson Iterative Method

Theorem 3.1 shows that Richardson iterative method is locally linear convergent in solving $\mathcal{M}$-Equations. In order to speed up the convergence rate of the Richardson iterative method, Anderson acceleration is considered in this section.

4.1 Anderson acceleration

Anderson acceleration is widely employed to enhance the convergence rate of the iterative methods in solving linear and nonlinear problems [16]. Unlike Richardson iteration, which only uses the previous point to approximate the solution at the current point, Anderson acceleration takes into account several previous points to minimize the residual. The number of the previous points is called the depth of Anderson acceleration.

Transforming the Richardson iteration into a fixed point problem $\mathbf{x} = \phi(\mathbf{x})$ in (16) leads to

$$\mathbf{r} = \mathbf{b} - \mathcal{A}\mathbf{x}^{m-1} = \frac{1}{\alpha}\left(\phi(\mathbf{x})^{[m-1]} - \mathbf{x}^{[m-1]}\right), \tag{34}$$

where $\mathbf{r}$ is the residual. Define

$$\mathbf{r}_k = \frac{1}{\alpha}\left(\phi\left(\mathbf{x}_k\right)^{[m-1]} - \mathbf{x}_k^{[m-1]}\right), \tag{35}$$

and

$$F_k = \left(\mathbf{r}_{k-l}, \mathbf{r}_{k-l+1}, \ldots, \mathbf{r}_k\right), \tag{36}$$

where l is the depth of Anderson acceleration. Let $\overline{\mathbf{x}}_k$ be the corrected value of $\mathbf{x}_k$ defined by,

$$\overline{\mathbf{x}}_k = \sum_{i=0}^{l} g_i^{(k)} \mathbf{x}_{k-l+i}, \tag{37}$$

where $\mathbf{g}_k = \left(g_0^{(k)}, g_1^{(k)}, \ldots, g_l^{(k)}\right)^T$ satisfies

$$\mathbf{g}_k = \arg\min_{\mathbf{g}_k \in \mathrm{R}^{l+1}} \|F_k \mathbf{g}_k\| \quad \text{subject to} \quad \sum_{i=0}^{l} g_i^{(k)} = 1. \tag{38}$$

Since $\sum_{i=0}^{l} g_i^{(k)} = 1$, then $g_l^{(k)} = 1 - \sum_{i=0}^{l-1} g_i^{(k)}$. Therefore, Equation (38) becomes an unconstrained equivalent problem,

$$\mathbf{g}_k = \arg\min_{\mathbf{g}_k \in \mathrm{R}^{l+1}} \left\| \mathbf{r}_k + \sum_{i=0}^{l-1} g_i^{(k)} \left(\mathbf{r}_{k-l+i} - \mathbf{r}_k\right) \right\|. \tag{39}$$

Let the elements of $\mathbf{g}_k$ be defined as,

$$g_i^{(k)} = \begin{cases} h_0, & i = 0, \\ h_i - h_{i-1}, & 1 \leqslant i < l, \\ 1 - h_l, & i = l, \end{cases} \tag{40}$$

and

$$h_i = \sum_{j=0}^{l} g_j, \qquad i = 0, 1, \ldots, l. \tag{41}$$

Let $R_k = \left[(\mathbf{r}_{k-l+1} - \mathbf{r}_{k-l}), \ldots, (\mathbf{r}_k - \mathbf{r}_{k-1})\right] \in \mathrm{R}^{n\times l}$. Then, Equations (38) and (39) are equal to

$$\mathbf{h}_k = \arg \min_{\mathbf{h}_k \in \mathrm{R}^{l+1}} \|\mathbf{r}_k - R_k \mathbf{h}_k\|_2. \tag{42}$$

Let $X_k = \left[(\mathbf{x}_{k-l+1} - \mathbf{x}_{k-l}), \ldots, (\mathbf{x}_k - \mathbf{x}_{k-1})\right] \in \mathrm{R}^{n\times l}$. Then, $\overline{\mathbf{x}}_k$ can be rewritten as

$$\begin{aligned} \overline{\mathbf{x}}_k &= \sum_{i=0}^{l} g_i^{(k)} \mathbf{x}_{k-l+i} \\ &= \mathbf{x}_k + \sum_{i=0}^{l-1} g_i^{(k)} \left(\mathbf{x}_{k-l+i} - \mathbf{x}_k\right) \\ &= \mathbf{x}_k - X_k \mathbf{h}_k. \end{aligned} \tag{43}$$

Thus, the Anderson acceleration can be regarded as a subspace method with the subspace of corrections spanned by F_k. Then, we can compute $\mathbf{x}_{k+1}$ via the standard Richardson iteration

$$\mathbf{x}_{k+1} = \phi\left(\overline{\mathbf{x}}_k\right). \tag{44}$$

4.2 Convergence analysis for Anderson Richardson method

In this section, we will prove that Anderson Richardson iteration converges faster than or equal to Richardson iteration method as stated in the following theorem. The algorithm for Anderson Richardson iterative method is given by Algorithm 2.

Theorem 4.1. *Consider an $\mathcal{M}$-Equation $\mathcal{A}\mathbf{x}^{m-1} = \mathbf{b}$ with $\mathbf{b} > \mathbf{0}$.If Richardson iteration is convergent, then the Anderson Richardson iteration converges faster than or equal to Richardson iteration method regardless of depth l.*

Proof. Let $\mathbf{r}_k$ and $\mathbf{x}_k$ be the kth residual and iteration, respectively, of the Richardson iteration. Similarly, let $\overline{\mathbf{r}}_k$ and $\overline{\mathbf{x}}_k$ be the kth residual and iteration of the Anderson Richardson iteration. According to Theorem 3.1, if Richardson iteration is convergent, then,

$$\frac{\|\mathbf{r}_k\|_2}{\|\mathbf{r}_{k-1}\|_2} < 1. \tag{45}$$

It follows from Equations (42) and (43) that,

$$\begin{aligned} \|\overline{\mathbf{r}}_k\|_2 &= \left\|\mathbf{b} - \mathcal{A}\overline{\mathbf{x}}_k^{m-1}\right\|_2 = \|\mathbf{b} - \mathcal{A}(\mathbf{x}_k - X_k\mathbf{h}_k)^{m-1}\|_2 \\ &\leqslant \|\mathbf{b} - \mathcal{A}\mathbf{x}_k^{m-1}\|_2 = \|\mathbf{r}_k\|_2 < \|\mathbf{r}_{k-1}\|_2. \end{aligned} \tag{46}$$

This indicates that Anderson Richardson iteration converges faster than or equal to Richardson iteration. □

Algorithm 2: Anderson Richardson Iterative Method

Input: Given an $\mathcal{M}$-tensor $\mathcal{A}$, a positive vector $\mathbf{b}$, an appropriate α, maximum iterative steps k_{max}, tolerance number ε and a positive initial vector $\mathbf{x}_0$. Let the deph for Anderson acceleration is l.

Output: $\mathbf{x}_k$

Set $k = 1$;
while $k < k_{\max}$, **do**
  Compute $\mathbf{r}_k = \mathbf{b} - \mathcal{A}\mathbf{x}_k^{m-1}$.
  Set $X_k = [(\mathbf{x}_{k-l+1} - \mathbf{x}_{k-l}), \ldots, (\mathbf{x}_k - \mathbf{x}_{k-1})] \in \mathrm{R}^{n\times l}$, $R_k = [(\mathbf{r}_{k-l+1} - \mathbf{r}_{k-l}), \ldots, (\mathbf{r}_k - \mathbf{r}_{k-1})] \in \mathrm{R}^{n\times l}$;
  if $k \mod l = 0$ **then**
    Determine $\mathbf{h}_k = \left[h_1^k, \ldots, h_l^k\right]$ such that $\mathbf{h}_k = \arg\min_{\mathbf{h}\in\mathrm{R}^l} \|\mathbf{r}_k - R_k\mathbf{h}_k\|_2$;
    Set $\overline{\mathbf{x}}_k = \mathbf{x}_k - X_k\mathbf{h}_k$;
    Set $\mathbf{x}_{k+1} = \left(\mathcal{I}\overline{\mathbf{x}}_k^{m-1} + \alpha\left(\mathbf{b} - \mathcal{A}\overline{\mathbf{x}}_k^{m-1}\right)\right)^{\left[\frac{1}{m-1}\right]}$;
  else
    $\mathbf{x}_{k+1} = \left(\mathcal{I}\mathbf{x}_k^{m-1} + \alpha\left(\mathbf{b} - \mathcal{A}\mathbf{x}_k^{m-1}\right)\right)^{\left[\frac{1}{m-1}\right]}$;
  if $||\mathcal{A}\mathbf{x}_{k+1} - \mathbf{b}|| \leqslant \epsilon$, **then**
    Output $\mathbf{x}_{k+1}$;
    Stop;
  else
    $k = k + 1$;
  Output "The method fails since it exceeds $k_{\max}$ iterations."

5 Preconditioned Richardson Iterative Method

Preconditioned techniques are widely used in accelerating iterative methods for linear and nonlinear systems. For multi-linear systems, several Preconditioned techniques have been proposed such as preconditioned Jacobian type method, preconditioned Gauss-Seidel type, and SOR type method [2, 9, 15]. Motivated by these works, we attempt to generalize Preconditioned techniques to Richardson iteration method in this section.

Multiplying both sides of multi-linear system (1) by a preconditioner tensor $\mathcal{P}$ yields,

$$\mathcal{P}\mathcal{A}\mathbf{x}^{m-1} = \mathcal{P}\mathbf{b}. \tag{47}$$

If the preconditioner is a nonsingular matrix instead of a tensor, then (47) becomes,

$$P\mathcal{A}\mathbf{x}^{m-1} = P\mathbf{b}, \tag{48}$$

where P is the preconditioner. Applying (48) on (9) leads to

$$\mathcal{I}\mathbf{x}^{m-1} = (\mathcal{I} - \alpha P\mathcal{A})\,\mathbf{x}^{m-1} + \alpha P\mathbf{b}, \tag{49}$$

where $\alpha P\mathcal{A}$ is a tensor and $\alpha P\mathbf{b}$ is a vector. By letting $\tilde{\mathcal{A}} = \alpha P\mathcal{A}$ and $\tilde{\mathbf{b}} = \alpha P\mathbf{b}$, (1) becomes,

$$\tilde{\mathcal{A}}\mathbf{x}^{m-1} = \tilde{\mathbf{b}}. \tag{50}$$

For strong $\mathcal{M}$-Equation with positive vector $\mathbf{b}$, Li et al. [9] introduced preconditioned Jacobi,

G-S, and SOR type methods using preconditioner $P_1 = I + S_\beta$, where,

$$S_\beta = \begin{pmatrix} 0 & -\beta_1 a_{12\ldots 2} & 0 & \ldots & 0 \\ 0 & 0 & -\beta_2 a_{23\ldots 3} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & -\beta_{n-1} a_{n-1,n\ldots n} \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix},$$

and $\beta_j \in [0,1]$, $j = 1, 2, \ldots, n-1$. Later, Liu et al. [15] improved Jacobi, G-S, and SOR type methods by proposing preconditioner $P_2 = I + R_\beta$, where,

$$R_\beta = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 \\ -\beta_1 a_{2,1\ldots 1} & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\beta_{n-2} a_{(n-1),1\ldots 1} & 0 & 0 & \ldots & 0 \\ -\beta_{n-1} a_{n,1\ldots 1} & 0 & 0 & \ldots & 0 \end{pmatrix},$$

and $\beta_j \in [0,1]$, $j = 1, 2, \ldots, n-1$. In the same year, Cui et al. [3] proposed another preconditioner $P_3 = I + T_\beta$ to improve the convergence rate of the same methods, where,

$$T_\beta = \begin{pmatrix} 0 & -\beta_1 a_{12\ldots 2} & -\beta_2 a_{1,3\ldots 3} & \ldots & -\beta_{n-1} a_{1,n\ldots n} \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix},$$

and $\beta_j \in [0,1]$, $j = 1, 2, \ldots, n-1$.

For the purpose of accelerating the rate of convergence of the Richardson iterative method developed in this paper, we employ the preconditioners P_1, P_2, and P_3 for strong $\mathcal{M}$-Equation with positive vector $\mathbf{b}$.

Lemma 5.1 and Theorem 5.1 below guarantees that the positive solution of system (1) remains the same even after applying preconditioners.

Lemma 5.1. *If tensor $\mathcal{A}$ is a strong $\mathcal{M}$-tensor, then $\alpha\mathcal{A}$ is also a strong $\mathcal{M}$-tensor for any $\alpha > 0$.*

Proof. Since $\mathcal{A}$ is a strong $\mathcal{M}$-tensor, then $\alpha\mathcal{A}$ is a $\mathcal{Z}$-tensor. According to Definition 2.6 and Lemma 2.1, if $\mathcal{A}$ is a strong $\mathcal{M}$-tensor, then there exists $\mathbf{x} \geqslant 0$ such that $\mathcal{A}\mathbf{x}^{m-1} > 0$. This implies that $\alpha\mathcal{A}\mathbf{x}^{m-1} > 0$ for any $\alpha > 0$. Thus, $\alpha\mathcal{A}$ is a strong $\mathcal{M}$-tensor. □

Theorem 5.1. *Let $\mathcal{A}$ an n dimensional mth-order strong $\mathcal{M}$-tensor and $\mathbf{b}$ be an n dimensional positive vector. Then the preconditioned system $\tilde{\mathcal{A}}\mathbf{x}^{m-1} = \tilde{\mathbf{b}}$ has the same unique positive solution with the original system $\mathcal{A}\mathbf{x}^{m-1} = \mathbf{b}$, where $\tilde{\mathcal{A}} = \alpha P\mathcal{A}$, $\tilde{\mathbf{b}} = \alpha P\mathbf{b}$ and $P = P_i$, $i = 1, 2, 3$ respectively.*

Proof. Li et al. [9], Liu et al. [15], and Cui et al. [3] have also proven that if $\mathcal{A}$ is a strong $\mathcal{M}$-tensor, then $P_k\mathcal{A}$, $k = 1, 2, 3$ are also strong $\mathcal{M}$-tensors. Hence, $\alpha P\mathcal{A}$ is also a strong $\mathcal{M}$-tensor. It is easy to see that if $\mathbf{b} > 0$ then $\alpha\mathbf{b} > 0$. By Lemma 2.5, $\tilde{\mathcal{A}}\mathbf{x}^{m-1} = \tilde{\mathbf{b}}$ and $\mathcal{A}\mathbf{x}^{m-1} = \mathbf{b}$ have the same positive solution. □

To show the convergence rate of preconditioned Richardson iterative method is faster than the original, the following Lemma 5.2 and Theorem 5.2 are necessary.

Lemma 5.2. *Let $\mathcal{A}$ be a strong $\mathcal{M}$-tensor and $\tilde{\mathcal{A}} = \alpha P_k \mathcal{A}$, $k = 1, 2, 3$, where $\alpha \in \left(0, \frac{1}{\max\limits_i a_{ii\cdots i}}\right]$ for all $\beta_j \in [0, 1]$, $j = 1, 2, \ldots, n-1$. Then, $\tilde{\mathcal{A}} = \mathcal{I} - (\mathcal{I} - \tilde{\mathcal{A}})$ is a regular splitting of $\tilde{\mathcal{A}}$.*

Proof. It can be seen that $\tilde{\mathcal{A}} = \mathcal{I} - (\mathcal{I} - \tilde{\mathcal{A}})$ is a splitting of $\tilde{\mathcal{A}}$. Based on Definition 2.7, since $M(\mathcal{I})$ is an identity matrix, $\mathcal{I}$ is left-nonsingular, and $M(\mathcal{I})^{-1} = I \geqslant O$, we only need to prove $\mathcal{I} - \tilde{\mathcal{A}} = \mathcal{I} - \alpha P\mathcal{A} \geqslant \mathcal{O}$. Because $\tilde{\mathcal{A}} = \alpha P\mathcal{A}$ is a strong $\mathcal{M}$-tensor, the off-diagonal elements of $\tilde{\mathcal{A}}$ is nonpositive. This indicates the off-diagonal elements of $\mathcal{I} - \tilde{\mathcal{A}}$ is nonnegative. Next, we consider the diagonal elements of $\mathcal{I} - \tilde{\mathcal{A}}$.

Case 1: $P_1 = I + S_\beta$.
Since $\tilde{\mathcal{A}} = \alpha P_1 \mathcal{A}$, then its elements are

$$\tilde{a}_{ji_2\ldots i_m} = \begin{cases} \alpha a_{ji_2\ldots i_m} - \beta_j \alpha a_{jj+1\ldots j+1} a_{j+1i_2\ldots i_m}, & 1 \leqslant j \leqslant n-1, \\ \alpha a_{ji_2\ldots i_m}, & j = n. \end{cases} \tag{51}$$

Given that $\alpha \in \left(0, \frac{1}{\max\limits_i a_{ii\cdots i}}\right]$, the diagonal elements $(j, i_2, \ldots, i_m) = (j, j, \ldots, j)$ are

$$\tilde{a}_{jj\ldots j} = \begin{cases} 1 - [\alpha a_{j,j\ldots j} - \beta_j \alpha a_{jj+1\ldots j+1} a_{j+1j\ldots j}] > 0, & 1 \leqslant j \leqslant n-1, \\ 1 - \alpha a_{nn\ldots n} \geqslant 0, & j = n. \end{cases} \tag{52}$$

Case 2: $P_2 = I + R_\beta$.
Similar to **Case 1**, the elements for $\tilde{\mathcal{A}} = \alpha P_2 \mathcal{A}$ are

$$\tilde{a}_{ji_2\ldots i_m} = \begin{cases} \alpha a_{ji_2\ldots i_m}, & j = 1,\ i_2, \ldots, i_m \in \langle n \rangle, \\ \alpha a_{ji_2\ldots i_m} - \alpha \beta_{j-1} a_{j1\ldots 1} a_{1i_2\ldots i_m}, & j \neq 1. \end{cases} \tag{53}$$

Therefore, the diagonal elements are

$$\tilde{a}_{jj\ldots j} = \begin{cases} 1 - \alpha a_{jj\ldots j}, & j = 1, \\ 1 - \alpha a_{jj\ldots j} + \alpha \beta_{j-1} a_{j1\ldots 1} a_{1j\ldots j}, & j \neq 1, \end{cases} \tag{54}$$

which indicates $1 - \tilde{a}_{jj\ldots j} \geqslant 0$.

Case 3: $P_3 = I + T_\beta$.
In this case, the elements for $\tilde{\mathcal{A}} = \alpha P_3 \mathcal{A}$ are

$$\tilde{a}_{ji_2\ldots i_m} = \begin{cases} \alpha a_{11\ldots 1} - \alpha \sum\limits_{j_2=2}^{n} \beta_{j_2-1} a_{j_2-1,j_2\ldots j_2} a_{j_2,i_2\ldots i_m}, & j = 1,\ i_2, \ldots, i_m \in \langle n \rangle, \\ \alpha a_{ji_2\ldots i_m}, & j = n,\ i_2, \ldots, i_m \in \langle n \rangle, \end{cases} \tag{55}$$

while its diagonal elements are given by

$$\tilde{a}_{jj\ldots j} = \begin{cases} \alpha a_{11\ldots 1} - \alpha \sum\limits_{j_2=2}^{n} \beta_{j_2-1} a_{j_2-1,j_2\ldots j_2} a_{j_2,1\ldots 1}, & j = 1, \\ \alpha a_{jj\ldots j}, & j \neq 1, \end{cases} \tag{56}$$

which implies that $1 - \tilde{a}_{jj\ldots j} \geqslant 0$.

Thus, based on the three cases above, we can conclude that $\mathcal{I}-\tilde{\mathcal{A}} \geqslant \mathcal{O}$, which means $\tilde{\mathcal{A}} = \mathcal{I}-(\mathcal{I}-\tilde{\mathcal{A}})$ is a regular splitting of $\tilde{\mathcal{A}}$. □

Theorem 5.2. *Let $\mathcal{A}$ be a strong $\mathcal{M}$-tensor and $\tilde{\mathcal{A}} = \alpha P_k \mathcal{A}$, $k = 1, 2, 3$ with $\alpha \in \left(0, \frac{1}{\max\limits_i a_{ii\cdots i}}\right]$ for all $\beta_i \in [0, 1]$, $i = 1, 2, \ldots, n-1$. Then, $\rho\left(\mathcal{I} - \tilde{\mathcal{A}}\right) < \rho\left(\mathcal{I} - \alpha\mathcal{A}\right) < 1$.*

Proof. Since $\alpha \in \left(0, \frac{1}{\max\limits_i a_{ii\cdots i}}\right]$, we have $\mathcal{I} - \alpha\mathcal{A} \geqslant \mathcal{O}$. According to Lemma 2.3, there exists $\lambda \geqslant 0$ and a nonnegative vector $\mathbf{y} \neq \mathbf{0}$ that satisfies $\lambda \leqslant \rho\left(\mathcal{I} - \alpha\mathcal{A}\right) < 1$, which leads to

$$\begin{aligned}\left(\mathcal{I} - \tilde{\mathcal{A}}\right)\mathbf{y}^{m-1} - \lambda\mathbf{y}^{[m-1]} &= \left(\mathcal{I} - \alpha P_k\mathcal{A}\right)\mathbf{y}^{m-1} - \lambda\mathbf{y}^{[m-1]} \\ &= (1-\lambda)\mathbf{y}^{[m-1]} - \alpha P_k\mathcal{A}\mathbf{y}^{m-1} \\ &= (1-\lambda)\mathbf{y}^{[m-1]} - P_k(1-\lambda)\mathbf{y}^{[m-1]} \\ &= \left(I - P_k\right)(1-\lambda)\mathbf{y}^{[m-1]},\end{aligned} \tag{57}$$

for $k = 1, 2, 3$. From (14) in Theorem 3.1 , we obtain $\lambda \leqslant \rho\left(\mathcal{I} - \alpha\mathcal{A}\right) < 1$.

Since $\lambda < 1$ and $\mathbf{y} \neq \mathbf{0}$, it follows from Equation (57) that $\left(\mathcal{I} - \tilde{\mathcal{A}}\right)\mathbf{y}^{m-1} < \lambda\mathbf{y}^{[m-1]}$. By Lemma 2.4, $\rho\left(\mathcal{I} - \tilde{\mathcal{A}}\right) < \lambda \leqslant \rho\left(\mathcal{I} - \alpha\mathcal{A}\right) < 1$. □

By Theorem 5.2, we can conclude that the preconditioned Richardson iteration converges faster than Richardson iterative method.

6 Numerical Examples

In this section, the numerical examples for Richardson iteration and its accelerations are presented. Numerical simulations to compare the Richardson iterative methods with SOR type method employed in [5], and preconditioned SOR type methods with three different preconditioners employed in [3], [9], and [15], respectively, are also carried out.

Example 6.1. *We construct a multi-linear system with third-order coefficient tensor similar to [5]. First, we generate a nonnegative third-order tensor $\mathcal{B}$ randomly using Matlab. Next, define scalar*

$$s = (1+\varepsilon)\cdot \max_{i=1,2,\ldots,n}(\mathcal{B}\mathbf{e}^2), \qquad \varepsilon > 0, \tag{58}$$

where $\mathbf{e} = (1, 1, \ldots, 1)^T$. By letting $n = 3$ and $\varepsilon = 0.05$, we can get an $\mathcal{M}$-tensor by setting $\mathcal{A} = s\mathcal{I} - \mathcal{B}$, where $\mathcal{A}$ along the first index is

$$A_{(1)} = \left(\begin{array}{ccc|ccc|ccc} a_{111} & a_{121} & a_{131} & a_{112} & a_{122} & a_{132} & a_{113} & a_{123} & a_{133} \\ a_{211} & a_{221} & a_{231} & a_{212} & a_{222} & a_{232} & a_{213} & a_{223} & a_{233} \\ a_{311} & a_{321} & a_{331} & a_{312} & a_{322} & a_{332} & a_{313} & a_{323} & a_{333} \end{array}\right)$$

$$= \left(\begin{array}{ccc|ccc|ccc} 5.3226 & -0.1107 & -0.4508 & -0.1107 & -0.9730 & -0.4224 & -0.4508 & -0.4224 & -0.0605 \\ -0.6820 & -0.5906 & -0.4754 & -0.5906 & 4.7333 & -0.6319 & -0.4754 & -0.6319 & -0.3993 \\ -0.0424 & -0.3077 & -0.4479 & -0.3077 & -0.8003 & -0.8143 & -0.4479 & -0.8143 & 4.8554 \end{array}\right).$$

To form a third order $\mathcal{M}$-Equation $\mathcal{A}\mathbf{x}^2 = \mathbf{b}$, the elements in the positive vector $\mathbf{b} = (9, 14, 13)^T$ are randomly generated from integer numbers 1 to 20 in Matlab. We select the parameter $\alpha = \dfrac{1}{\max\limits_{i\in<n>} a_{ii\cdots i}}$ based on (26) in Theorem 3.3. Next, set the initial vector $\mathbf{x}_0 = \mathbf{e} = (1, 1, \ldots, 1)^T$ for all iterations in this example. Based on the Anderson acceleration simulation results for $2 \leqslant l \leqslant 5$, $l = 3$ produces the fastest convergence rate. Therefore, we set the depth $l = 3$ by experiment.

The relationships between the number of iterative steps and relative residuals are shown in Figure 2. It demonstrates that the Richardson iterative methods with and without preconditioner are linearly convergent, while the Richardson with Anderson acceleration method is nonlinear. These numerical results confirm Theorem 3.1.

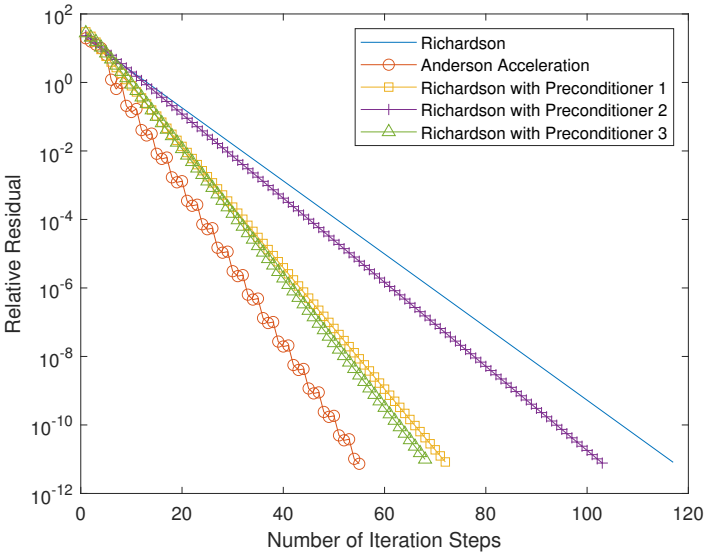


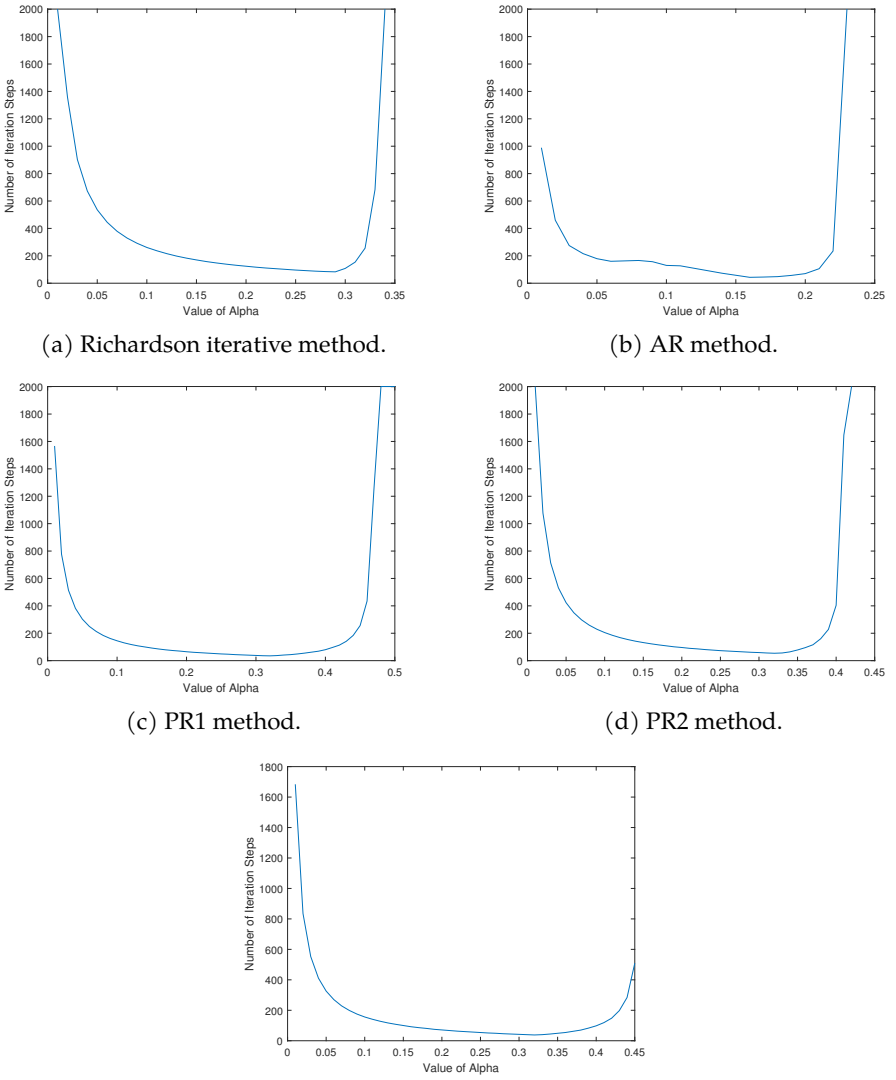
Figure 2: Relationship between the number of iteration steps and relative residual for five different iterative methods in solving Example 6.1.

Table 1 shows the comparison of five different Richardson iterative methods for solving the third order $\mathcal{M}$-Equation. It is observed that introducing the Anderson acceleration and preconditioners into Richardson iterative method enhances the convergence rate. Note that different preconditioner produces different convergence rate. For the convenience of representation, abbreviations are given to denote the methods.

Table 1: Comparison of five different Richardson iterative methods for solving Example 6.1.

Iterative Methods	Abbreviations of the Methods	Iterative Steps	CPU(s)
Richardson	Richardson	117	0.0011
Richardson with Anderson acceleration	AR	55	0.0007856
Richardson with Preconditioner P_1	PR1	72	0.0006387
Richardson with Preconditioner P_2	PR2	103	0.0008672
Richardson with Preconditioner P_3	PR3	68	0.0006813

To determine the values of α that produce the least iterations for each iterative methods, we set α from 0.1 to 1 with step size 0.01. Relationship between the number of iterations and relative residuals for different iterative methods is shown in Figure 3.



(a) Richardson iterative method.

(b) AR method.

(c) PR1 method.

(d) PR2 method.

(e) PR3 method.

Figure 3: Relationship between the number of iterations and relative residuals for five different Richardson iterative methods.

The number of iterations(Iter) and CPU(s) for five different Richardson iterative methods in Figure 3, where $\alpha \in [0.01, 0.5]$ with step size 0.01 are displayed in Table 2. Since all the methods are not convergent when $\alpha > 0.5$, we only show the results for $\alpha \leqslant 0.5$. "null" in the table means the method is not convergent at the given α value. From the table, the convergence ranges of α for each method are evident. For example, the convergence range of α for Richardson iterative method is $[0.01, 0.33]$. Furthermore, the smallest number of iterative steps and CPU time are in bold, which implies the corresponding value of α is optimal in the simulation.

Therefore, the results in Table 2 indicate that each iterative method has different values of

optimal α with different ranges. Similar to the results in Table 1, AR, PR1, PR2, and PR3 can reduce the number of iterative steps compare to Richardson iterative method.

Table 2: Comparison of five different Richardson iterative methods in terms of number of iterations and CPU(s) for $\alpha \in [0.01, 0.5]$ with step size 0.01.

	Richardson		AR		PR1		PR2		PR3	
α	Iter	CPU(s)	Iter	CPU(s)	Iter	CPU(s)	Iter	CPU(s)	Iter	CPU(s)
0.01	2001	0.0140	1007	0.0615	1754	0.0259	2001	0.0198	1684	0.0179
0.02	1359	0.0084	460	0.0074	871	0.0087	1077	0.0109	835	0.0083
0.03	902	0.0058	278	0.0044	576	0.0053	714	0.0069	553	0.0051
0.04	673	0.0039	218	0.0035	429	0.0037	532	0.0045	411	0.0034
0.05	536	0.0031	179	0.0048	340	0.0037	423	0.0039	326	0.003
0.06	445	0.0028	160	0.0021	281	0.0024	350	0.0029	270	0.0022
0.07	379	0.0024	163	0.0023	239	0.002	298	0.0024	229	0.0019
0.08	330	0.0026	169	0.0024	208	0.0019	259	0.0022	199	0.0016
0.09	292	0.0020	157	0.002	183	0.0017	229	0.0019	175	0.0015
0.10	261	0.0018	130	0.0019	163	0.0014	205	0.0018	156	0.0013
0.11	237	0.0016	127	0.0017	147	0.0012	185	0.0018	141	0.0015
0.12	216	0.0015	109	0.0068	134	0.0011	168	0.0016	128	0.0011
0.13	198	0.0014	91	0.0012	123	0.001	154	0.0015	117	0.0009758
0.14	183	0.0014	73	0.0009405	113	0.0009484	142	0.0013	108	0.0008958
0.15	170	0.0013	58	0.0007446	104	0.0008739	132	0.0012	100	0.0008372
0.16	158	0.0012	**43**	**0.0005397**	97	0.0008233	123	0.0011	92	0.0007705
0.17	148	0.0011	45	0.0005632	90	0.0008145	115	0.0009661	86	0.0007242
0.18	139	0.0011	48	0.0005995	85	0.0007242	108	0.0009927	81	0.000719
0.19	131	0.0010	57	0.0007152	79	0.0006662	101	0.0008612	75	0.000629
0.20	124	0.0009445	70	0.0008802	75	0.0006247	96	0.000838	71	0.0005945
0.21	117	0.0008886	106	0.0015	70	0.0006642	90	0.0007717	67	0.0005608
0.22	111	0.0008422	235	0.0029	66	0.0005598	86	0.0007365	63	0.0005212
0.23	106	0.0008042	null	null	63	0.0005267	81	0.0006998	60	0.0004964
0.24	101	0.0007689	null	null	60	0.0005016	77	0.0006587	57	0.0004749
0.25	96	0.0007329	null	null	57	0.0004804	73	0.0006202	54	0.0004445
0.26	92	0.000703	null	null	54	0.000577	70	0.0005924	51	0.0004201
0.27	88	0.0006757	null	null	**51**	0.0004424	67	0.0005663	49	0.000409
0.28	85	0.0006767	null	null	52	**0.0004335**	64	0.000571	46	0.0003794
0.29	**83**	**0.0006371**	null	null	55	0.0005142	61	0.0005188	44	0.0003618
0.30	108	0.0008144	null	null	61	0.00052	59	0.0004896	42	0.000345
0.31	154	0.0014	null	null	68	0.0005653	56	0.0004626	40	0.0003296
0.32	256	0.0020	null	null	76	0.0006387	**54**	**0.0004461**	**38**	**0.0003138**
0.33	683	0.0050	null	null	86	0.0007246	56	0.0006561	40	0.0003592
0.34	null	null	null	null	100	0.0008385	64	0.0006717	44	0.0003717
0.35	null	null	null	null	117	0.0009718	78	0.0007456	49	0.0004008
0.36	null	null	null	null	143	0.0012	95	0.0009116	54	0.0004441
0.37	null	null	null	null	182	0.0015	116	0.001	62	0.0005082
0.38	null	null	null	null	250	0.002	160	0.0013	70	0.0005769
0.39	null	null	null	null	392	0.0031	228	0.002	83	0.0006987
0.40	null	null	null	null	864	0.0073	404	0.0034	98	0.0008402
0.41	null	null	null	null	null	null	1647	0.0134	119	0.0009705
0.42	null	null	null	null	null	null	null	null	148	0.0012
0.43	null	null	null	null	null	null	null	null	198	0.0016
0.44	null	null	null	null	null	null	null	null	284	0.0022
0.45	null	null	null	null	null	null	null	null	509	0.0039
0.46	null	null	null	null	null	null	null	null	2192	0.0182
0.47	null	null	null	null	null	null	null	null	null	null
0.48	null	null	null	null	null	null	null	null	null	null
0.49	null	null	null	null	null	null	null	null	null	null
0.50	null	null	null	null	null	null	null	null	null	null

In order to test and verify the results in Theorem 3.1 and Theorem 3.2, in Table 3, we compare

the values of α that produce the least iterations in Table 2 with the optimal α calculated by (18) in Theorem 3.2 for Richardson and preconditioned Richardson iterative methods (PR1, PR2 and PR3), as well as the range of α for convergence in Theorem 3.1.

Table 3: Comparison of theoretical calculation and numerical test for different Richardson iterative methods at optimal α and range of α.

Iterative Methods	λ_1	λ_2	Theoretical α_{opt}	Optimal α in Table 2	Theoretical range of α	Range of α in Table 2
Richardson	1.0311	5.9474	0.2866	0.29	(0, 0.3363)	[0.01, 0.33]
PR1	1.6265	3.8141 + 2.0325i	0.2729	0.27	(0, 0.4084)	[0.01, 0.40]
PR2	1.3105	4.7914 + 0.4740i	0.3243	0.32	(0, 0.4134)	[0.01, 0.41]
PR3	1.7123	4.1077 + 0.9324i	0.3235	0.32	(0, 0.4630)	[0.01, 0.46]

In Table 3, λ_1, λ_2 and theoretical α_{opt} are calculated using (18) and (19) in Theorem 3.2, while the theoretical range of α is calculated using (13) in Theorem 3.1. The values in Table 3 clearly demonstrate that the calculated theoretical α_{opt} and range of α coincide with the numerical results in Table 2.

Figure 4 shows the dynamics of Richardson iterative method visually for three different initial vectors $\mathbf{x}_{0,1} = (1, 1, 1)^T$, $\mathbf{x}_{0,2} = (4.9, 7.3, 6)^T$ and $\mathbf{x}_{0,3} = (5.8, 7.1, 6.3)^T$. It can be seen that the iterations converge from the different initial vectors to the exact solution $\mathbf{x}_*$.

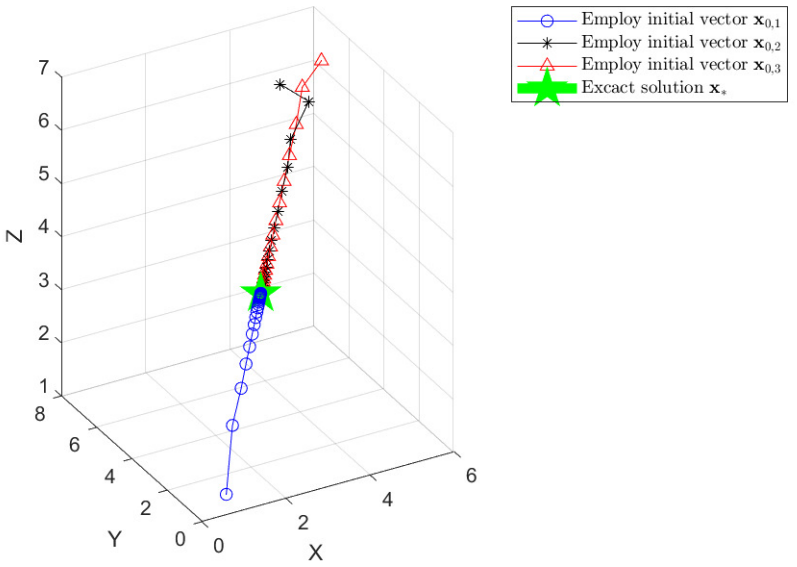


Figure 4: Dynamics of Richardson iterative method for three different initial vectors.

Example 6.2. *Consider the same problem as Example 6.1. We compare the proposed five Richardson iterative methods to SOR type method [5] and preconditioned SOR type methods with three different preconditioners in [3], [9], and [15], respectively. The SOR type method involved splitting $\mathcal{A} = \mathcal{D} - \mathcal{L} - \mathcal{F}$, where $\mathcal{D}$ is a diagonal tensor with $\mathcal{D} = D\mathcal{I}$, $\mathcal{L}$ is a tensor with $\mathcal{L} = L\mathcal{I}$, and D, L are the diagonal and strictly lower triangle part of $M(\mathcal{A})$. Then we can get the iterative scheme for SOR type methods:*

$$\mathbf{x}_k = M\left(\frac{1}{\omega}(\mathcal{D} - \omega\mathcal{L})\right)^{-1}\left((\mathcal{F} - \omega\mathcal{I})\mathbf{x}_{k-1}^{m-1} + \mathbf{b}\right)^{[\frac{1}{m-1}]}, \quad k = 1, 2, \ldots. \tag{59}$$

Following [3], [9], and [15], we set $\omega = 1$ for SOR type methods and the α_{opt} shown in Table 3 for the five Richardson iterative methods. The numerical results are reported in Table 4.

Table 4: Comparison of five different Richardson iterative methods and four different SOR type methods.

Iterative Methods	Iterative Steps	CPU(s)
Richardson	82	0.0006371
AR	**43**	**0.0005397**
SOR type	108	0.0016
PR1	**50**	**0.0004356**
SOR type with Preconditioner P_1	62	0.0005636
PR2	**53**	**0.0004532**
SOR type with Preconditioner P_2	100	0.0009832
PR3	**38**	**0.0003134**
SOR type with Preconditioner P_3	56	0.0006118

According to Table 4, it can be concluded that Richardson iterative methods with optimal α is superior to SOR type methods with $\omega = 1$ in terms of iteration steps and time, both for with or without preconditioner.

Example 6.3. *Consider a similar problem to Example 6.1 but with higher dimension $\mathcal{M}$-Equation by setting $\mathcal{A} \in \mathbb{R}^{[3,10]}$. The construction of $\mathcal{M}$-Equation and the analysis are similar to the previous Example 6.1, which produce the results in Table 5 and Figure 5.*

Table 5: Comparison of five different Richardson iterative methods for solving Example 6.3.

Iterative Methods	Iterative Steps	CPU(s)
Richardson	239	0.0142
AR	167	0.0108
PR1	171	0.0115
PR2	185	0.0088
PR3	95	0.0034

Based on Table 5 and Figure 5, we observe similar result to Example 6.1, where Richardson iterative method with Anderson acceleration and preconditioners converge faster than the one without.

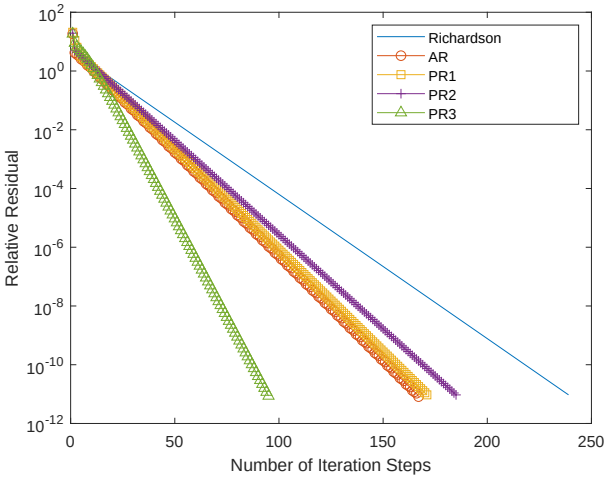


Figure 5: Relationship between the number of iteration steps and relative residual for five different Richardson iterative methods in solving Example 6.3.

7 Conclusions

In this paper, five Richardson iterative methods for solving $\mathcal{M}$-Equation have been proposed. The convergence of these methods has been proven theoretically and confirmed numerically. By adding Anderson acceleration and preconditioners to the Richardson iterative method, the convergence rate can be improved. We also theoretically determined the optimal parameter α and the convergence range of α for Richardson iterative method and show that they coincide with the numerical simulations. Also, the range of initial vector for iteration is extended from the range in past study. Furthermore, by comparing the proposed five Richardson iterative methods to the four SOR type methods in previous studies, we show that Richardson iterative methods with optimal α perform better in terms of number of iterative steps and CPU time.

However, among these five Richardson iterative methods, we cannot single out the best method for solving different $\mathcal{M}$-Equations since AR performs the best convergence rate in Example 6.1 while PR3 is the best in Example 6.3. For future work, one can consider which method is the best for some special kind of $\mathcal{M}$-Equation and investigating other type of accelerators that produce better convergence rate in solving $\mathcal{M}$-Equation.

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Conflicts of Interest The authors declare no conflict of interest.

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